Embodied design for spatial geometry learning

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16/12/2022

האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

Seminar on Mathematics Education, Institute of Mathematics, Pedagogical University of Cracow

- Palatnik, A. & Abrahamson, D. (2022). Escape from Plato's cave: An enactivist argument for learning 3D geometry by constructing tangible models. In G. Bolondi, F. Ferretti, & C. Spagnolo (Eds.), *Proceedings of the Twelfth Congress of the ERME (CERME12).* Bolzano, Italy: ERME.
- Palatnik, A. (2022). Students' exploration of tangible geometric models: Focus on shifts of attention. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 275-282). PME.
- Benally, J., Palatnik, A., Ryokai, K. & Abrahamson, D. (2022). Charting our embodied territories: Learning geometry as negotiating perspectival complementarities. *For the Learning of Mathematics*. *42*(3), 34-41.
- *Rosenski, D. & Palatnik, A. (2022). Secondary students' experience using 3D pen in spatial geometry: affective states while problem solving. In G. Bolondi, F. Ferretti, & C. Spagnolo (Eds.), Proceedings of the Twelfth Congress of the Proceedings of the Twelfth Congress of the ERME (CERME12).* Bolzano, Italy: ERME*). Bolzano, Italy: ERME.*

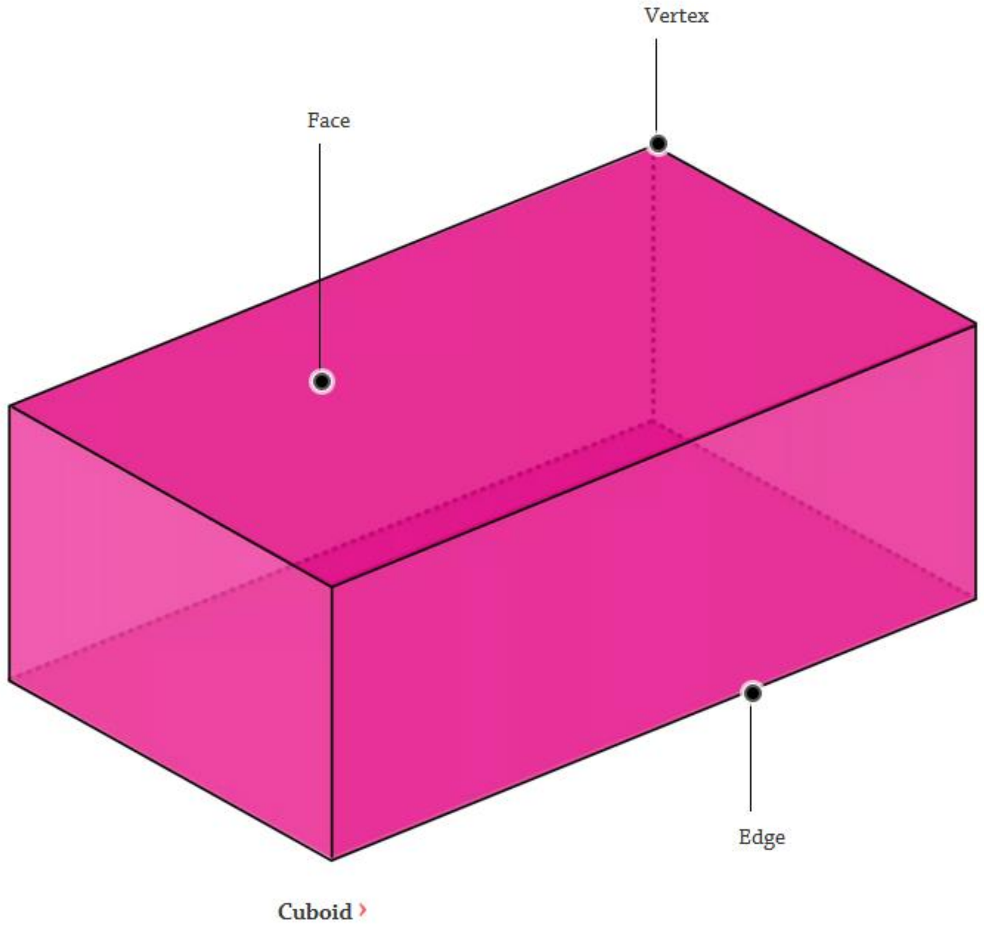
Complementary studies

- Palatnik, A., & Sigler, A. (2019). Focusing attention on auxiliary lines when introduced into geometric problems. *International Journal of Mathematical Education in Science and Technology*, *50*(2), 202-215. https://doi.org/10.1080/0020739X.2018.1489076.
- Palatnik, A. & Sigler, A. (2021, July). *Introduction of an auxiliary element as a shift of attention*. Presented at Topic Study Group 9 "Teaching and learning of geometry at secondary level", 14th International Congress of Mathematics Education (ICME-14) July, 2021, Shanghai, China.

“And then my student asked me, do the boxes really exist?”

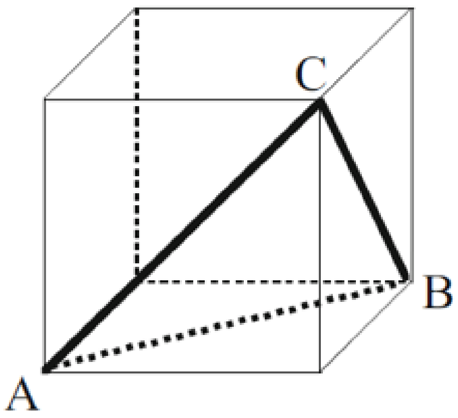
A citation from a reflection of the preservice teacher assignment

Palatnik, A., & Cohen-Eliahu, N. (2022). "My world is no longer as narrow as the ant world": the development of knowledge for teaching mathematics: Development of knowledge for teaching mathematics in the teacher education program..



The Hebrew word for a cuboid (or a rectangular parallelepiped) is תֵּבָה teva.

Challenging for students



In a cube, can you identify the shape ABC?

Choose your answer from (a) – (e).

(a) Right-angled triangle

(b) Isosceles triangle.

(c) Right-angled isosceles triangle

(d) Equilateral triangle

(e) Scalene triangle

	3-(2)
G4 ($N=261$)	12.6
G5 ($N=213$)	16.9
G6 ($N=209$)	36.8
G7 ($N=225$)	34.2
G8 ($N=224$)	35.3
G9 ($N=225$)	52.4

Fujita, T., Kondo, Y., Kumakura, H., Kunimune, S., & Jones, K. (2020). Spatial reasoning skills about 2D representations of 3D geometrical shapes in grades 4 to 9. *Mathematics Education Research Journal*, *32*, 235-255.

Spatial geometry in schools

- Students are struggling with spatial geometry tasks (Fujita et al., 2020).
- Students are required to visualize three-dimensional objects given two-dimensional drawings. (Widder et al., 2019)
- Are we giving them all the tools for success?

Spatial geometry

- A combination of spatial vision and geometric knowledge is required
- There is not much knowledge about the processes of solving tasks in 3D.
- Mithalal and Balacheff (2019) continue Duval (1998) from working with figures to the perception of geometric properties.
- Dimensional, mereological and instrumental deconstruction, perhaps shifts of attention?

Molyneux's problem

- William Molyneux initially presented this question to John Locke in 1688. Locke then interposed the question within the Second edition of his An Essay Concerning Human Understanding:
- “Suppose a Man born blind, and now adult, and taught by his touch to distinguish between a Cube, and a Sphere of the same metal, and nighly of the same bigness, so as to tell, when he felt one and t’other, which is the Cube, which the Sphere. Suppose then the Cube and Sphere placed on a Table, and the Blind Man to be made to see. Quære, Whether by his sight, before he touched them, he could now distinguish, and tell, which is the Globe, which the Cube” (Locke, 1694/1975, p.146).

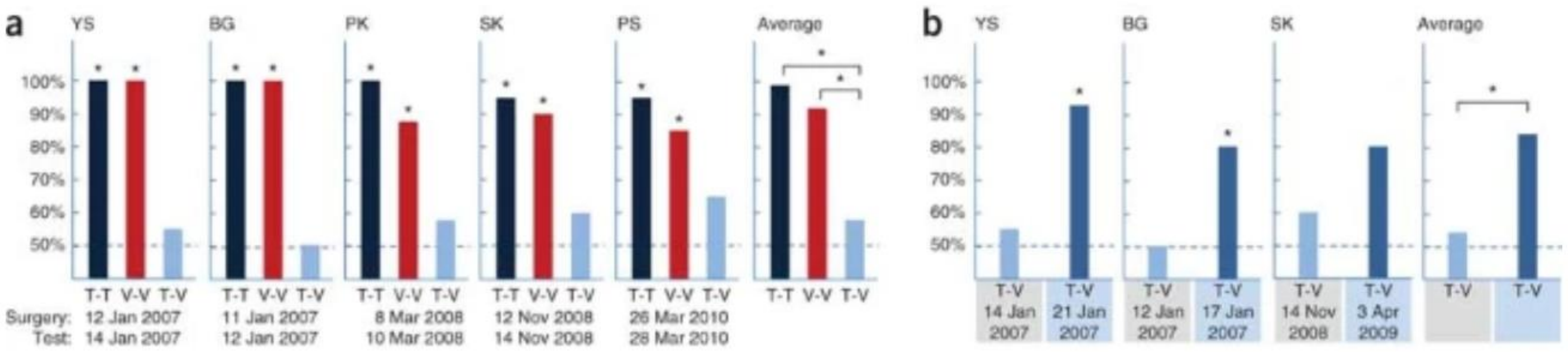
Molyneux's problem

- The answer of Locke is negative: "the blind man, at first sight, would not be able with certainty to say which was the globe, which the cube, whilst he only saw them" (p.146).
- Both Molyneux and Locke suggested that the reason for the negative answer is that the connections between our ideas of figure acquired through different sense modalities must be learned through experience: “For, though he has obtained the experience of how a globe, how a cube affects his touch, yet he has not yet obtained the experience, that what affects his touch so or so, must affect his sight so or so; or that a protuberant angle in the cube, that pressed his hand unequally, shall appear to his eye as it does in the cube” (p.146).

Molyneux's problem

Held, R., Ostrovsky, Y., de Gelder, B., Gandhi, T., Ganesh, S., Mathur, U., & Sinha, P. (2011). The newly sighted fail to match seen with felt. *Nature neuroscience*, *14*(5), 551-553.

Figure 2: Intra- and inter-modal matching results.



Does our way of teaching make students kinesthetically and tactile blind?

How we can change that?

Embodied turn in cognitive scinces

- Perception and action are formatively constitutive of our thinking—cognition is modal and situated activity (e.g., Chemero, 2013)
- The mind's function is not to *represent* the environment precisely but to *engage* with it dynamically vis-à-vis socio–biological task demands and emergent contextual contingencies.
- The environment offers opportunities for potential action—affordances (Gibson, 1986/ 2014)—that the agent interactively discerns and incorporates.
- When we engage the world with fellow humans, we coordinate with them perceptual orientations in relation to shared situations

Foundations of design principles

- Embodied learning
  - Enactive and embodied cognition. Perception and action are formatively constitutive of our thinking—cognition is modal and situated activity (e.g., Chemero, 2013)
  - Distributed cognition. “reflective conversation with material” (Bamberger and Schön,1983) where rearrangements of material objects support problem solving
  - Embodied design for mathematics education. Conceptual learning, could emanate from, or be triggered by, experiences of enacting or witnessing conceptually oriented movement forms, and only then formalization of these gestures and actions in disciplinary formats and language (Abrahamson et al., 2020)

Foundations of design principles

- Papert's constructionism (Papert, 1980; Papert & Harel, 1991)
  - Learning through making with an emphasis on learners' interactions with their artifacts
  - The critical role of the cultural surrounding whilst building internal cognitive structures
- Gibson's ecological psychology
  - The environment offers opportunities for potential action—affordances (Gibson, 1986)—that the agent interactively discerns and incorporates
- Learning mathematics with physical models and machines
  - Legacy of Leonardo da Vinci and Felix Klein
  - "A model—whether it be executed and looked at, or only vividly presented—is not a means for this geometry, but the thing itself" (Klein 1893, p. 42).[1]

The common features of the activities

- Spatial geometry tasks;
- Students generate and transform material artifacts
- First construct, then answer questions
- Collaborative problem-solving
- Different scale of the models

- The instruction contains written text and two-dimensional diagrams

From research goals to RQs

- How students ground geometry concepts in action-oriented perception?
- Which role student multimodal action-based interactions with concrete material plays when students solve spatial geometry tasks?
- When students study 3D geometrical objects by exploring physical models, which shifts of attention do they experience?
- What role do physical features of the models (i.e., their relative size and their orientation in space) play in the process of student exploration?

Shifts of attention (Mason 1989, 2008, 2010)

- Learning is a transformation of attention involving “shifts in the form as well as the focus of attention” (Mason, 2010,p. 24).
- *What* is attended to and *how* the objects are attended
- Five different forms or *structures of attention.*
  - One may *hold the wholes* without focusing on particularities
  - *Discern details* among the rest of the elements of the attended object.
  - One may *recognize relationships* between discerned elements
  - *Perceive properties* by actively searching for additional elements fitting the relationship.
  - *Reasoning based on perceived properties*.
- The shifts in attention structures are not necessarily sequential, and one may return to *holding the whole* to reassess the situation.

Method. The context and the participants

- An outdoor implementation of the activity.
- A group of six tenth-grade students.
- Semiformal. A part of an enrichment program for the students before their first year in the new high school.

Research goals

- Design of learning environments for geometry instruction in accordance with the principles of the theory of embodied learning
- Enrichment of the theory on the basis of analysis of learning that takes place in these environments.
  - Characterization of cognitive processes involved in solution of spatial geometry tasks, especially student breakthroughs
  - Characterization of students interaction with media
  - Characterization of students collaboration
  - Characterization of emotional engagement of students

From research goals to questions

How students ground geometry concepts in action-oriented perception?

Which role played student multimodal action-based interactions with concrete material when students solved spatial geometry tasks?

Three teaching experiments

- Students' experiences with 3D pen sketching while solving spatial geometry problem. Three 10th grade students faced the task for approximately 20 minutes.
- Students experiences of construction and exploration of human-scale models of 3D solids. Four 7th grade students constructed the model and then worked on several questions about its geometric properties (40 minutes)
- Students experiences of construction and exploration of human-scale and hand-held models of 3D solids. Four 9th grade students constructed the models and then worked on several questions about its geometric properties (40 minutes)

Method. The context

- How many vertices does the polyhedron you built have? How did you calculate that?

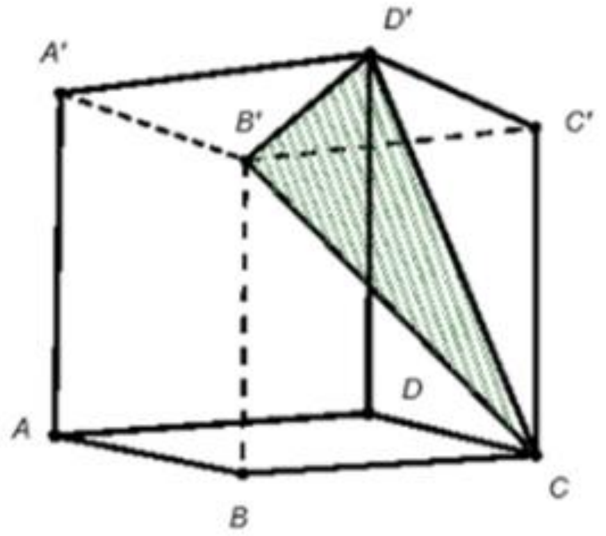
- How many edges does the polyhedron you built have? How did you calculate that?

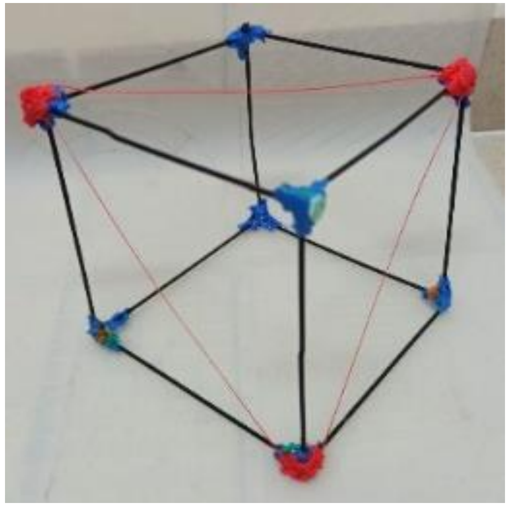
Method (data collection and analysis)

- Video-audio recording
- Combination of multimodal analysis of students' interactions (Abrahamson et al., 2020) with microgenetic analysis of shifts of attention (Voutsina et al., 2019)
  - Transcription of the activity, overlaid with a description of students' actions, gestures, and movements
  - Division of the activity into episodes (e.g. coping with the first question)
  - Search for indicators of the shifts in *focus* and *structures of attention*
  - Comparing students interactions within the team and with models by means of actions, gestures, movements and speech in different episodes

TE 1: Learning a 3D model's affordances for spatial problem solving

- 2D diagram
- 3D cube model
- 3D pen sketching

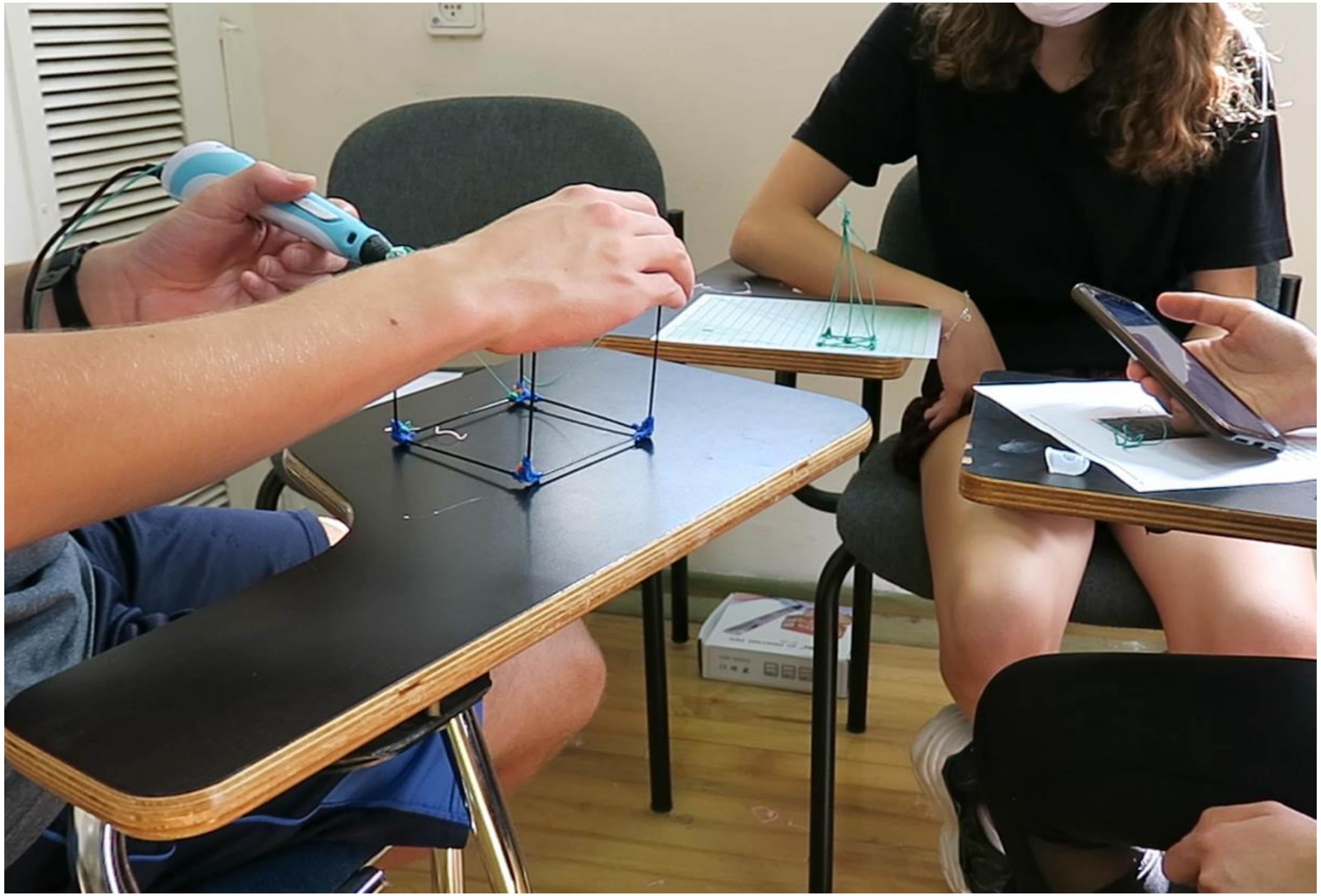


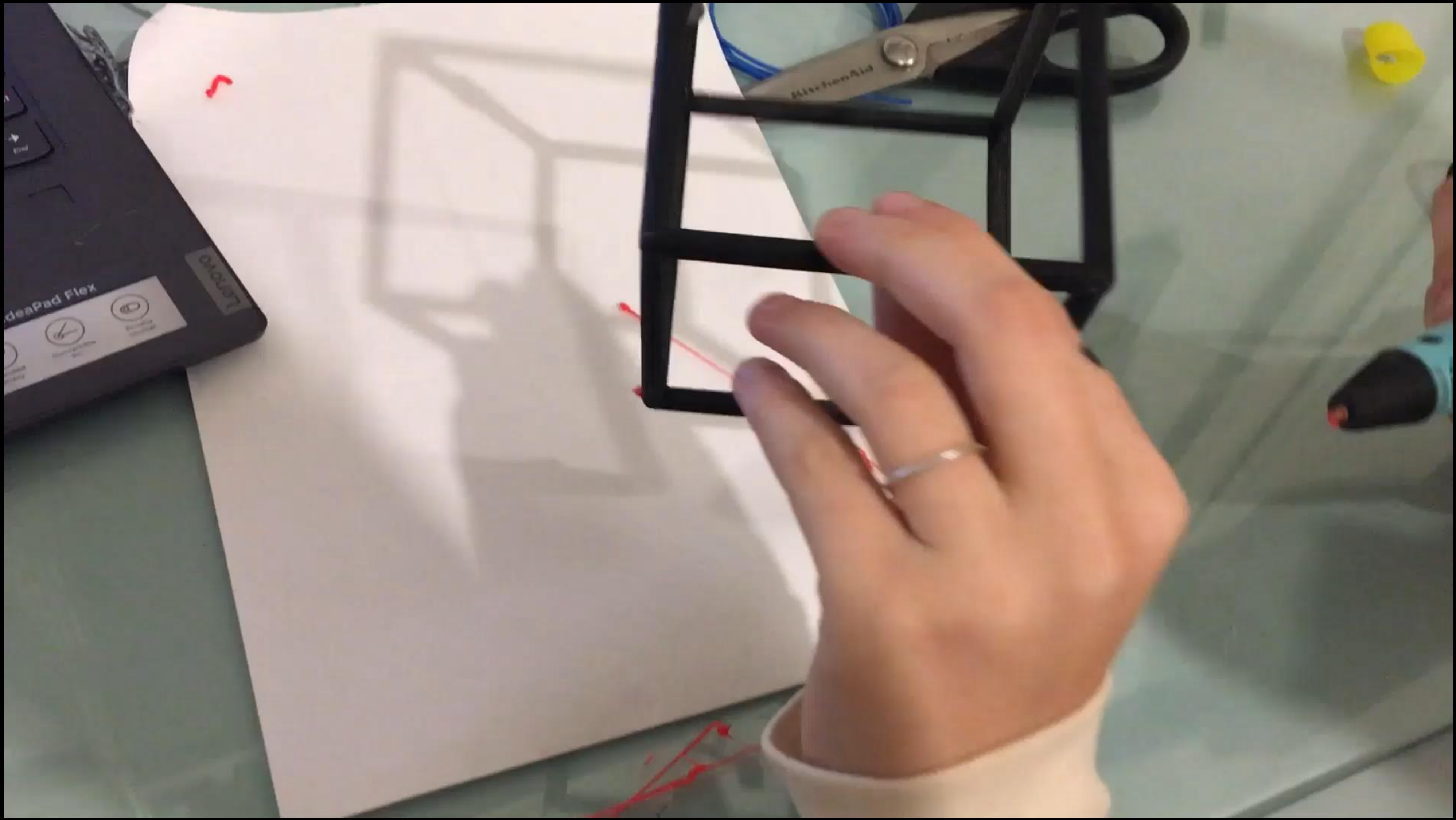


ABCDA'B'C'D' is a cube. Answer the true/ false questions and explain your reasoning.

1. CB'D' is a right-angled triangle. The right angle is ___?
2. B'D' is the shortest side of the triangle CB'D'.
3. Triangle CB'D' has an obtuse angle.
4. In triangle CB'D', all angles are equal.

3D sketching




IdeaPad Flex
Lenovo
KitchenAid

T: No, but you know that this is a cube, and this (cube face) is a square, and then, this is 90 degrees, then this is 90 degrees, and then it (diagonal) bisects the angle, so it is 45(degrees).

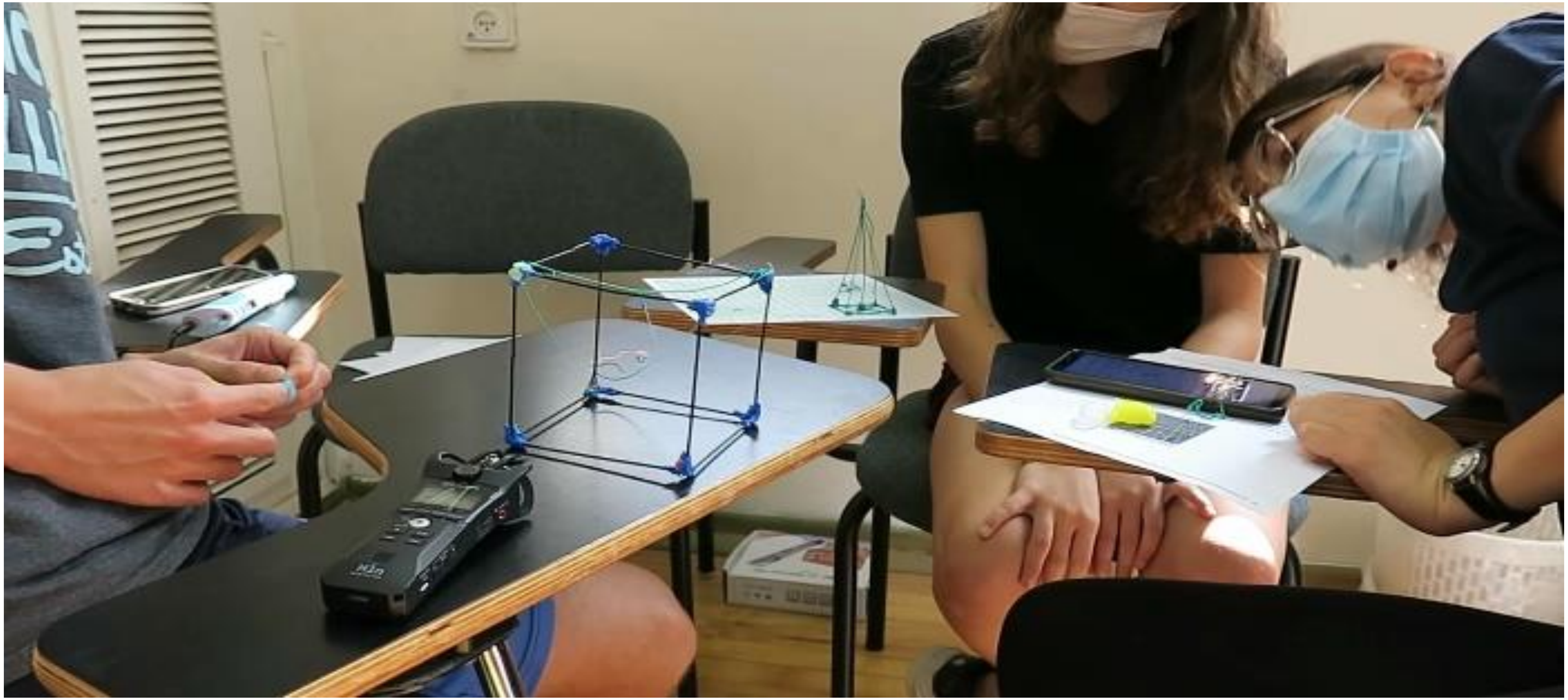
G: The question is, if we rotate (two faces, where the edge is an axis) [gestures "rotation" with two palms as faces], would it be the same angle? Could it be?

M: To them, it (the angle) in the picture also looks like that (90 degrees).

T: In the picture, it is just from a different angle, if you turn it like this [adjusts the model], you can see that this [points] is the right angle if this is stretched [pulls up the slightly sagging plastic diagonal of the top face of the cube.]

The students made inferences and conjectures about the task without taking the model in hand, only lightly touching it and making minor adjustments. Most of these adjustments reoriented the 3D model vis-à-vis the given 2D diagram.

Shifts of Perspective



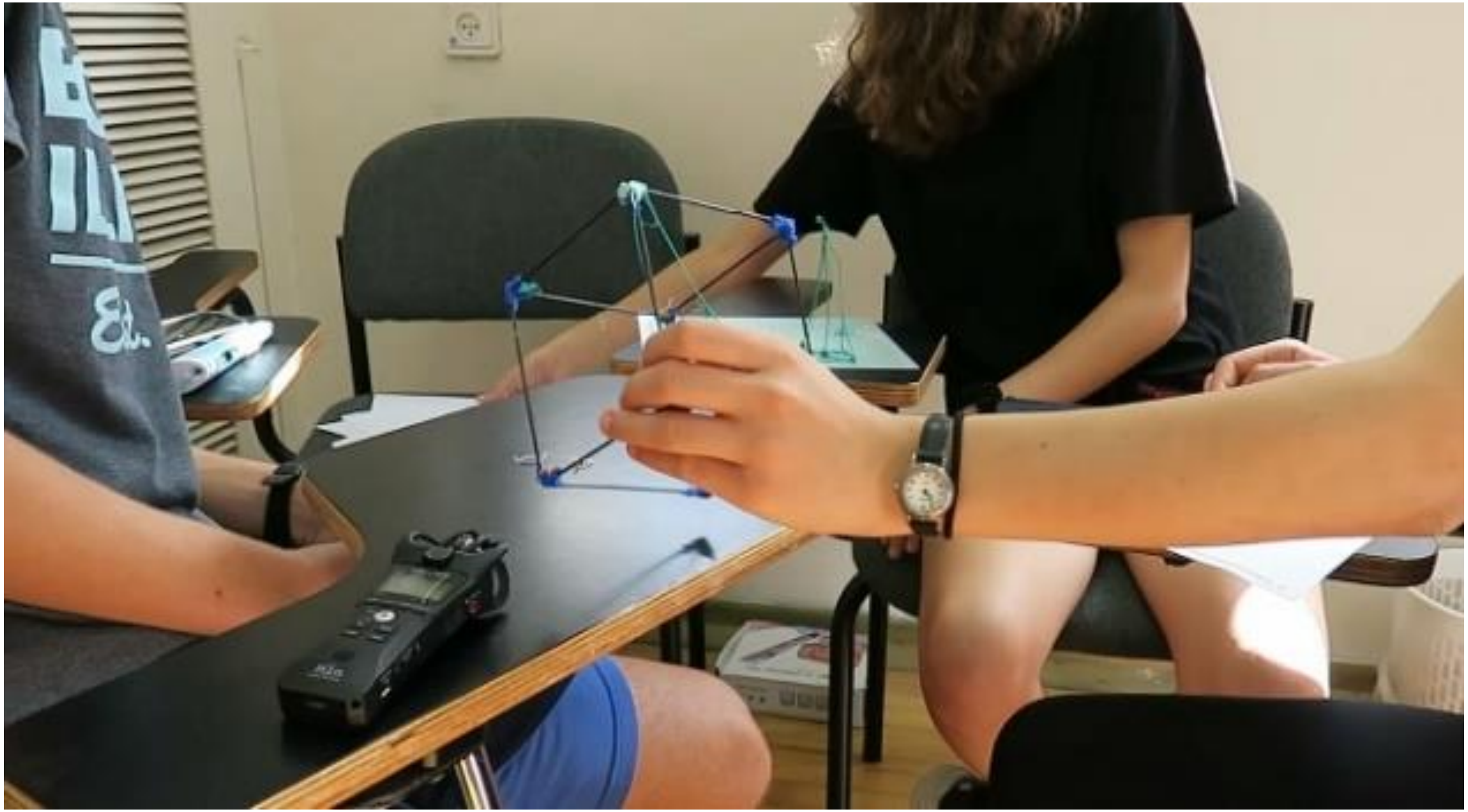
Towards the breakthrough

G was unsatisfied with the claim that CB’D’ is a right-angled triangle.

G: “If these two angles look like the same shape, if they are both right angles and a triangle has a total of 180 degrees, then the triangle cannot exist”;

“This is impossible. If you rotate it each way to make it (one of the angles) look like a right angle, then you can rotate it in a different way (to make other angle look like a right angle)”.

The student took matters into her hands



Changing the way of interaction with a model

T took the model in hand for the first time, rotated it for five seconds, and 20 seconds later said, "Now I get it."

T tried to formulate her vision, accompanying her explanation with more than 5 complex model rotations (i.e., more than 2 axes involved).

The core of her argument was in line with what G had previously argued: from different angles, the triangle appears isosceles, and yet its angle "can't be 90 for all of them."

During this minute, G's attention was on the rotating model, enabling him to see it under these transformations.

In contrast, M focused only on the 2D diagram and tried several times to draw her peers' attention to it: "Look at the picture!"

T: So you think all the angles are 60 degrees? Is it an equilateral triangle?

G: Yeah, look, all the sides are of the same length [traces with finger three sides in succession] if you look at it. Is that true?

T: Mmmm…I don’t know.

G: Look [adjusts the model], all the sides (of the triangle) are exactly a diagonal [traces a diagonal of the upper face], the diagonal of a square. All the squares are the same. And that means, if all of the sides are equal [points on a different face], all of the angles are equal, and they’re all 60 degrees.

T: Ok... [types into the response form] We think that all sides are equal; therefore, the angles are also equal... equal to 60. It’s an equilateral triangle.

Findings- TE 1

- The task demanded that students harmonize their spatial reasoning skills with domain-specific knowledge of planar geometry (c.f. Fujita et al. 2020)
- Coordination of traditional and novel medium is not easy for students
  - 3D drawing per se was not enough for the solution
  - Even given a model and after actually drawing the problem the solution was not straight forward
  - Students tentatively experimented with their new degrees of modal freedom—looking at objects, pointing at them, touching them, lifting and rotating them
  - Rotation of the model combined with geometric conjecture led to the collaborative breakthrough
  - In tasks combining 2D diagrams and 3D models students may over rely on traditional medium and 2D representation
- Different points of view on the same model provided fruitful for students advancement

Findings from the research on 3D sketching

- Great variation between students in the way they interact with 3D models when solving spatial geometry questions accompanied by 3D pen.
- It takes time for students to get used to a 3D medium, overcoming dependence on 2D
- Breakthroughs in the solution are related to movement, rotation, change of perspective or understanding of the other's point of view (literally)

Rosenski, D. & Palatnik, A. (2021, February). *To feel the sketch*. Paper presented at the 9th Jerusalem Conference on Research in Mathematics Education–JCRME9 (in Hebrew).

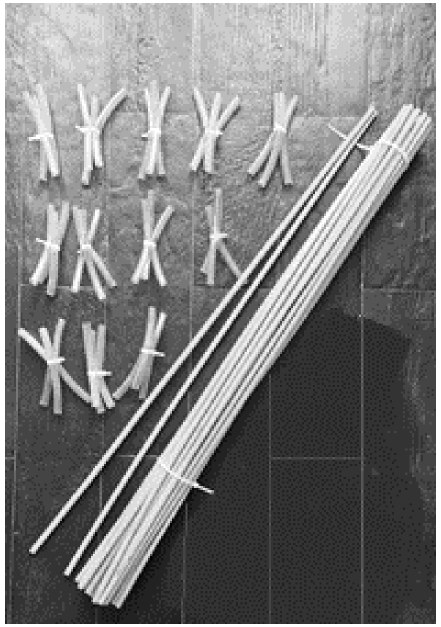
Rosenski, D. & Palatnik, A. (2022). Secondary students' experience using 3D pen in spatial geometry: affective states while problem solving. In G. Bolondi, F. Ferretti, & C. Spagnolo (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12, February 6 – 10, 2022).* Bolzano, Italy: ERME.

Palatnik, A. & Abrahamson, D. (2022). Escape from Plato's cave: An enactivist argument for learning 3D geometry by constructing tangible models. In G. Bolondi, F. Ferretti, & C. Spagnolo (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12, February 6 – 10, 2022).* Bolzano, Italy: ERME.

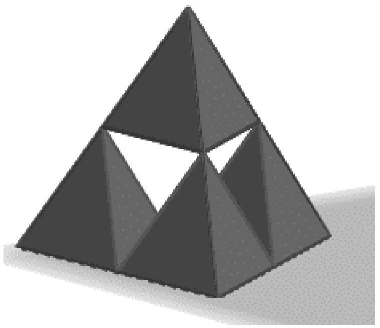
TE 2: Constructing enactive argumentation—gesture, action, medium

1) Your team has to construct a three-dimensional model of the following geometrical solid using a construction kit. The solid has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converge at each vertex.

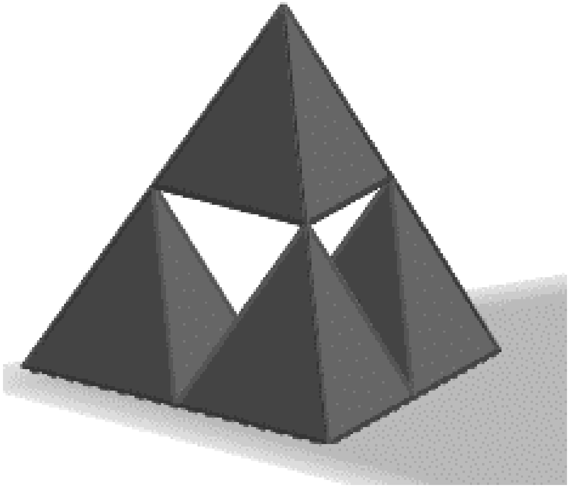


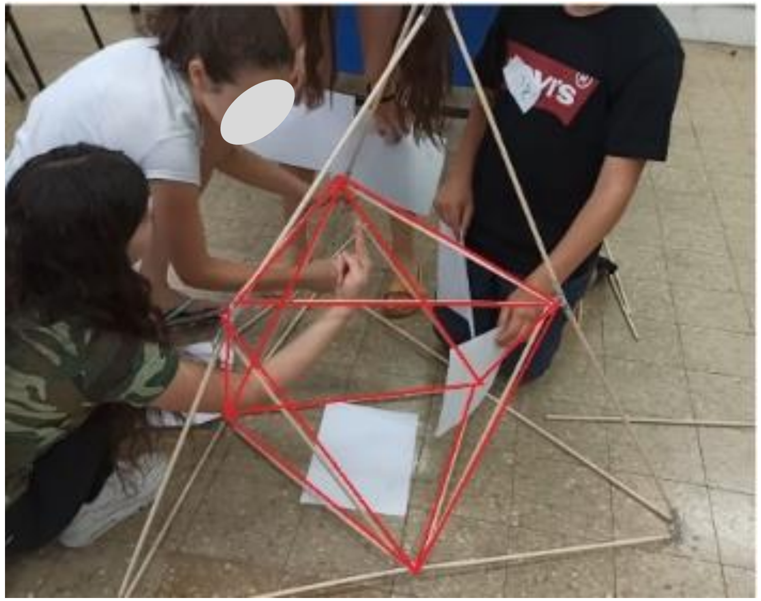
2) The polyhedron you've constructed is called a tetrahedron. Construct a similar polyhedron whose edges are 2 times larger than the original one. You can use the image below for construction.

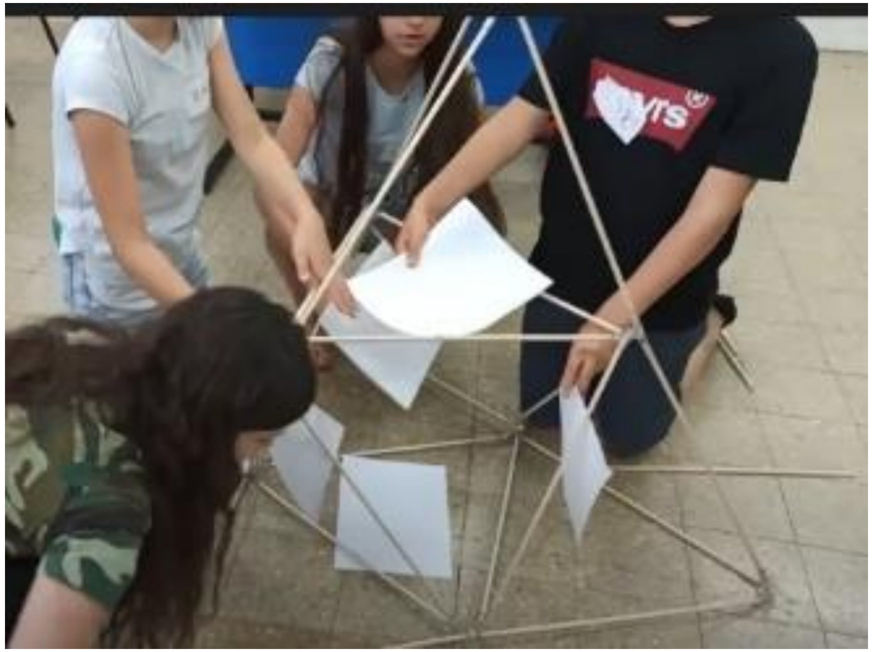


TE 2: Further steps. Constructing enactive argumentation—gesture, action, medium

- “Comparing the volumes of the large and small tetrahedra that you built, how many times the volume of the large tetrahedron is greater? Explain your answer.
- Several small tetrahedra compose the large tetrahedron. Can you describe a three-dimensional shape between them? Can you construct it?”



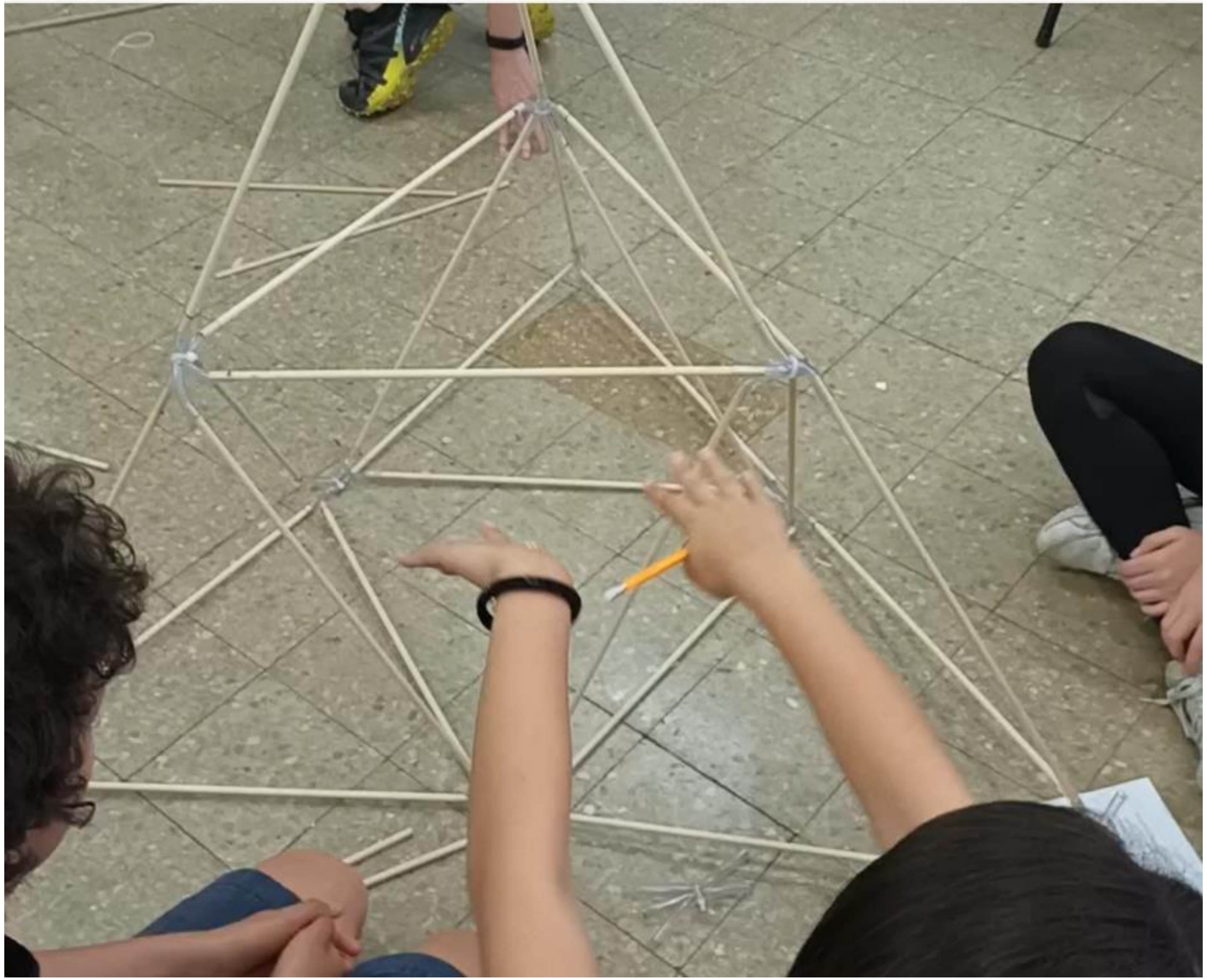




Once the group had constructed the first small pyramid, Yali placed it on his head.

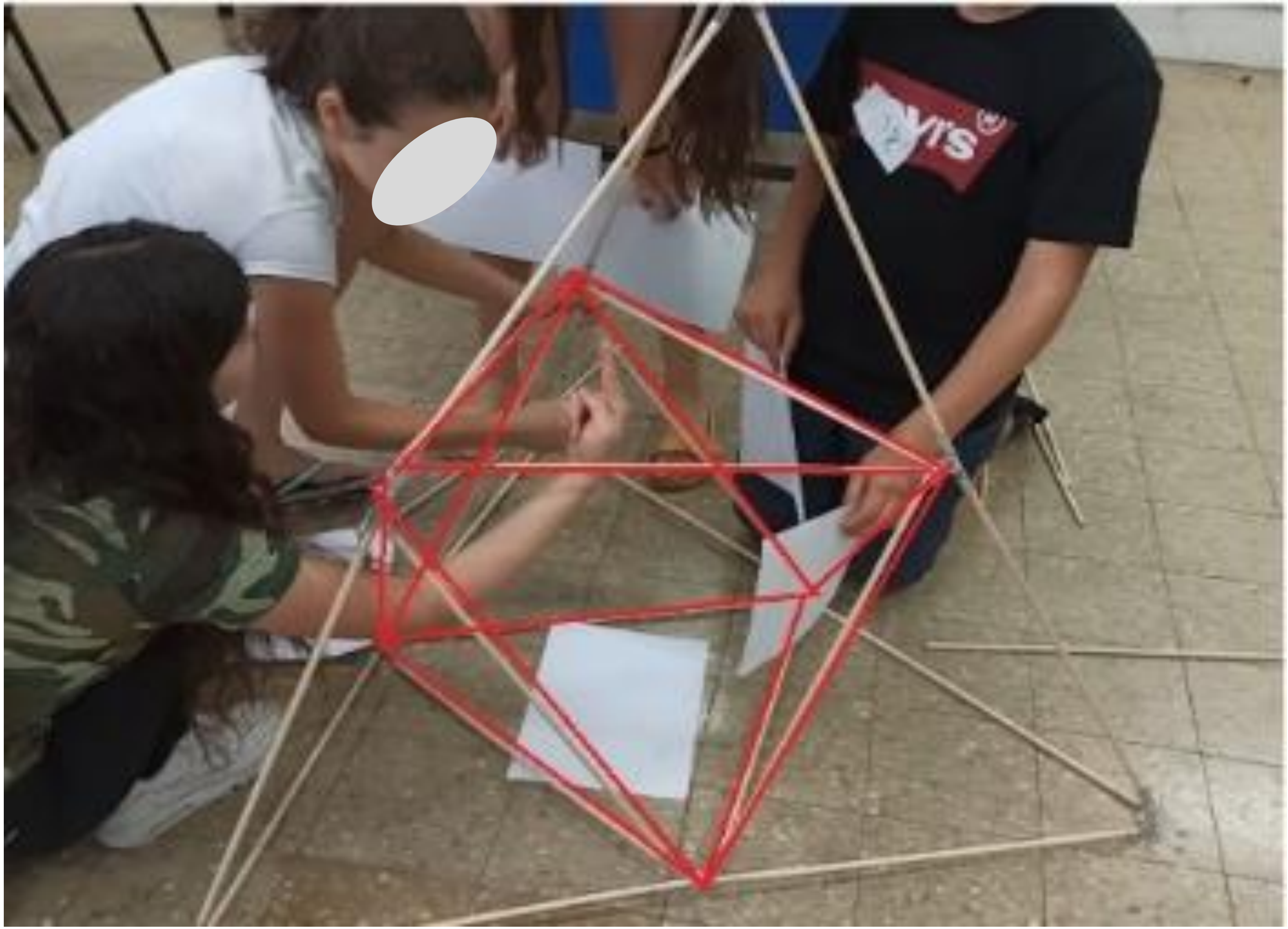
Nami, using Yali as a stand, gestured on him that this polyhedron is called “arba-on” (“arba” is four in Hebrew). The palms of her hands present the polyhedron’s faces.

Then, removing the model off Yali’s head, Nami gestured similarly, though with her forearms, to present the same four faces.



It took the students approximately 11 minutes of collaborative work to construct a large model and begin answering the items. Their plan for estimating the large tetrahedron’s volume was to decompose it into its component parts.

They easily recognized four small tetrahedra: “three at the base, and one at the top.” However, the students were not sure about the shape of a three-dimensional hollow between the tetrahedra (the octahedron).


vi's

Tami, Nami, and Gali suggested that the hollow is also shaped as a tetrahedron. Yali disagreed and offered to count the faces of the "empty space."

He rotated the large model, hoping to render it more familiar, yet that action proved unhelpful.

Tami remonstrated, "You just can't *see* this (tetrahedron)." To support her claim, she grabbed two sheets of paper lying on the desk and applied them successively as the polyhedron's faces, expecting these to total at four.

Immediately, Yali appropriated Tami’s strategy, just to disprove her.

Summoning more paper sheets and distributing them over more group members, he marshaled an “octopus of hands” to simultaneously cover all the polyhedron’s faces.

The introduction of these auxiliary objects helped students to solidify the shape, count the faces, and eventually write the following definition: “The polyhedron between four triangular (pyramids) has eight identical faces. Each face is an equilateral triangle.”

Findings- TE 2

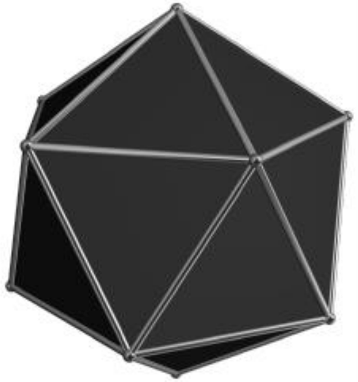
- The need to define the object raised naturally from the idea to compute the volume as sum of volumes
- Students' solution was collaborative:
  - Adoption and adaptation of each others ideas
  - Collaborative actions and gestures
- The solution involved different types of gestures: pointing, formative and iconic gestures; actions, use of available media
- The solution involved the transition from iconic visualization to non-iconic visualization was carried out by introducing tangible auxiliary elements (paper sheets in the form of polyhedron faces) into the 3D model
- The object was defined by stating the property of the faces and by enumerating them

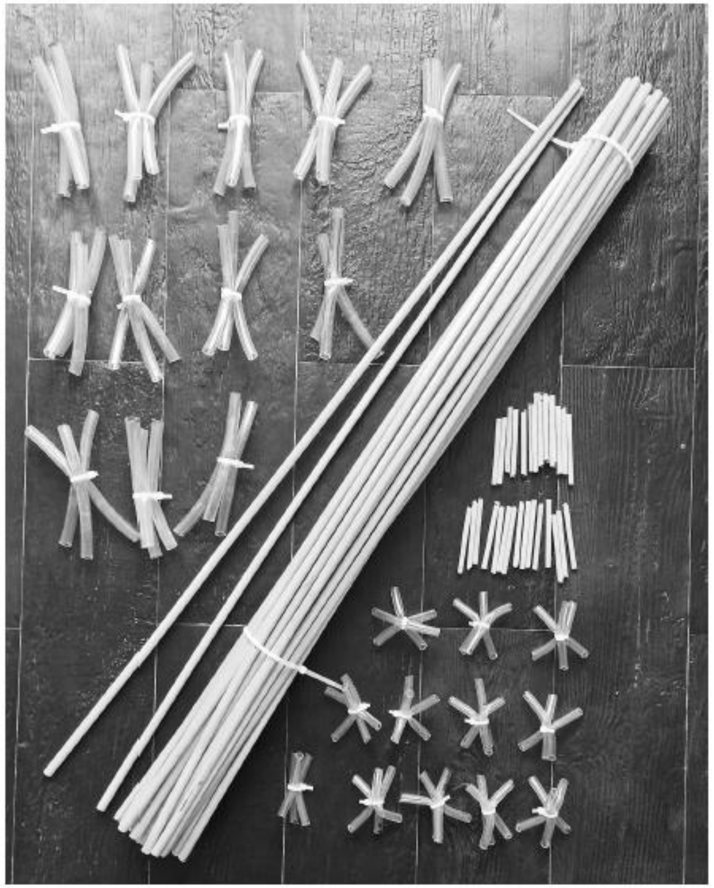
TE 3: Education of perception through shifts of attention

Your team has to construct two three-dimensional models (one large, one small) of a geometric solid, a polyhedron.

The polyhedron has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converge at each vertex.





The questions

A. How many edges does the polyhedron you built have? How did you calculate that?

B. If the polyhedron was half full with water, what would be the shape of the surface of the water when the polyhedron is on a triangular base? What about when the polyhedron is tilted onto a single vertex?

C. What other questions can be invented about this polyhedron?

Foundations of design principles for TE-3

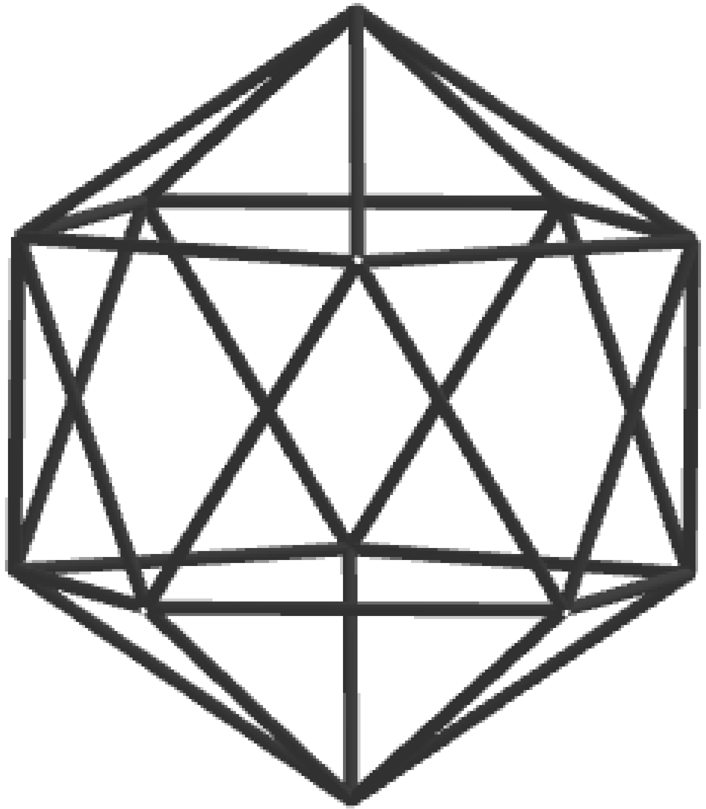
- Mathematical modeling of geometric figures should take into account four distinct perceptual systems of the figure(s) (e.g. Herbst et al., 2017):
  - (a) as physical navigation of macrospace (objects more than 50 times the size of an individual);
  - (b) as capturing an object in mesospace (0.5 to 50 times);
  - (c) as constructions of small objects in microspace (less than 0.5 times);
  - (d) as descriptions and manipulations of small objects in microspace

Findings

- Episode 1 finding the number of vertices
- Episode 2: finding the number of edges

Students discuss a small-scale model resting on a triangular face

Students discuss a small-scale model resting on a vertex



Grey	[holds a model on its vertex, starts counting] One, two, three, four, five. [touches an upper vertex, touches two additional vertices]
Black	How many vertices? [Reaches out for the model and touches it]. We already counted (them). It is a number of joints.
Yellow	[takes the model, starts to count by touching silicon joints] One, two.
Grey	Twelve. Times five. Sixty.
Yellow	How (it can be) twelve times five? How (it can be) sixty? [looks at the model].
Grey	[tries to take a model from Yellow] Ah, vertices… Twelve. Put it (the model) like this [tries to orient the model on the vertex]
Yellow	Give it (the model) to me for a moment. I know what I'm doing [takes the model away from Gray].
Grey	But, but…It's… Ohhh…
Yellow	[starts counting the joints from two facing her]. One, two. [continues counting] One, two, three…ten. It is twelve! [puts a model on a floor to write an answer]
Grey	[takes a model and tilts it on a vertex] Look [addressing Yellow] at it this way. [Starts counting from a lower basis] One, two…
Yellow	There are twelve!!!

Intermediate summary

- The small model is the focus of students attention
- Small model affords the students:
  - to group around it
  - to simultaneously grasp most of its features, *holding the whole*.
  - All the vertices are in the hand reach enabling students to *discern* them by touching
- The counting was conducted in two different ways
  - By orienting the model in a particular way Grey *recognized a relationship* between several groups of vertices of the icosahedron
  - Yellow was also successful in her attempt to count the vertices, which she separated into two groups of two and ten and did not see the value in the alternative orientation of the model in space.

Findings

Finding the number of edges

Yellow: How many edges are there?

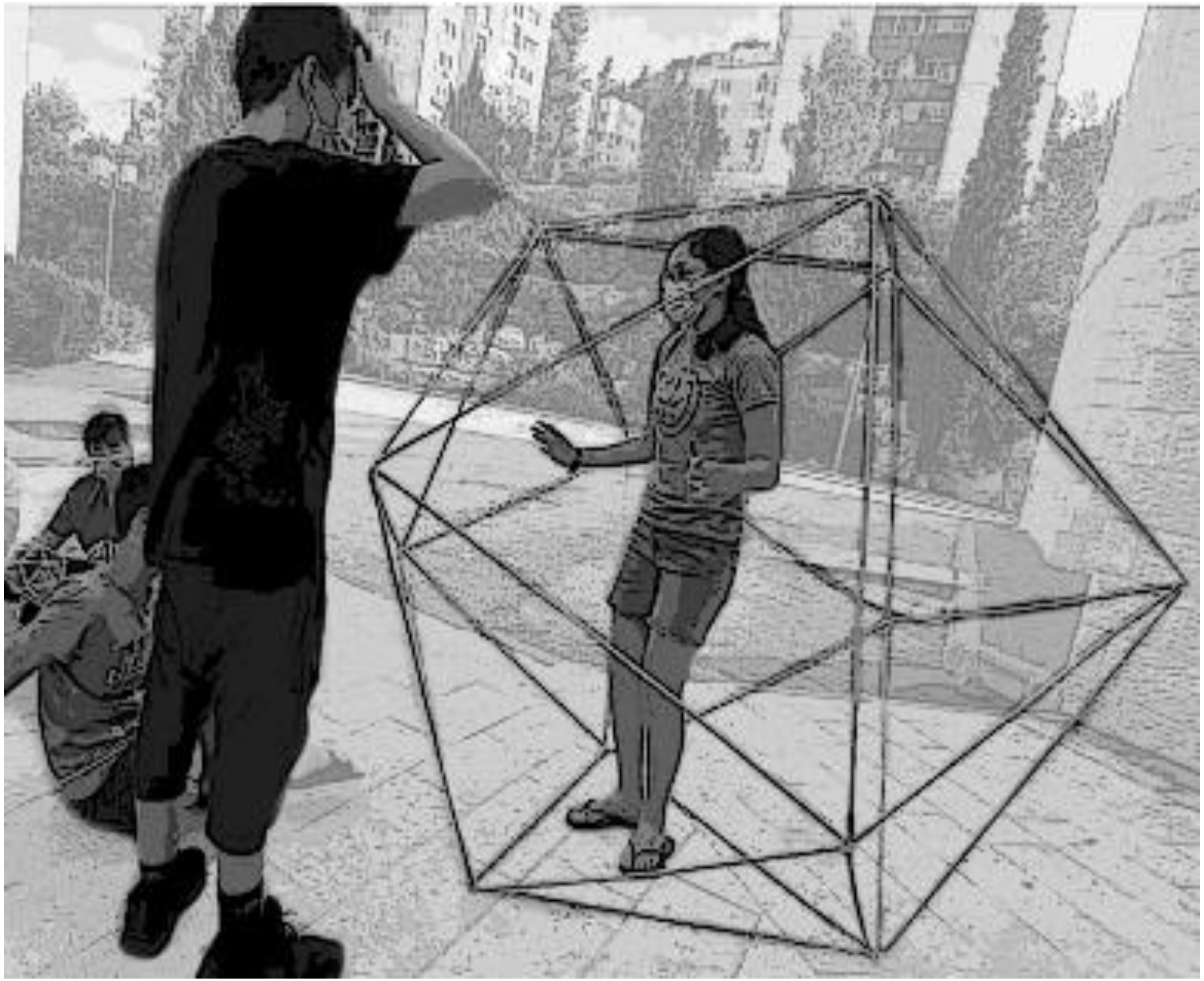
Black : Okay, that’s tricky because they’re shared. (i.e., each edge is shared by two triangles).

Blue: I’ll put a finger [on the first edge, to Yellow help her monitor the count].

Orange: You just count the sticks.

Yellow: I’ll go to the big one (i.e., the large-scale model).

Black : The big one is just nicer.

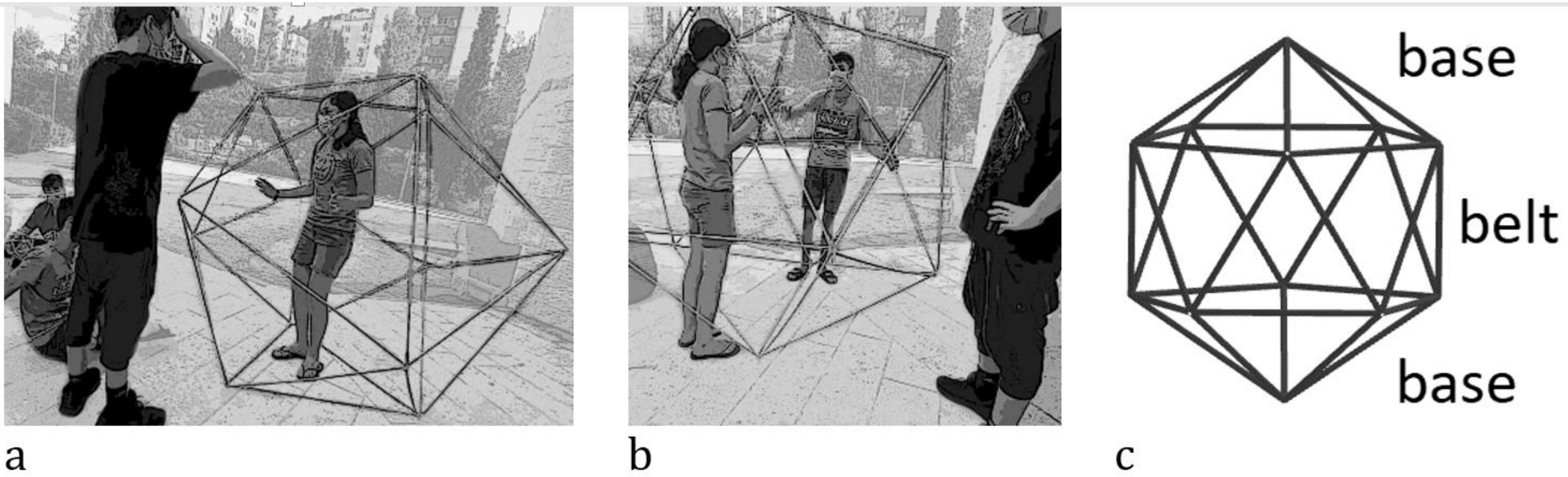


Finding the number of edges (continues)

Yellow	There are five from each vertex. One should be subtracted. Then there are four. Two should be subtracted here. It's three. It doesn't work that way…3, 4, 5 [sits on the floor, inside the model, frustrated]. I can't count this. [Stands up]. How many sticks did we use [during the construction stage]? Three and another three, and another three, and another three, it's 12…
Grey	Let's do it as we did with (inaudible) [Stands up]
Yellow	[referring to triangular faces] …another three, 15, another three…
Black	We need a formula for this…
Gray	I'm tilting it. [starts tilting the model]
Yellow	No, no, no, no! eighteen…No! Why?

Findings

Finding the number of edges



a) students' problem-solving attempt inside and outside a human-scale model (standing on a triangular face); (b) having tilted the structure onto a vertex, the students soon arrive at a critical breakthrough; (c) partition of an icosahedron

Finding the number of edges (continues)

Gray	To make it like this (standing on the vertex). It will be easier to count like that [holds the model on the vertex] (Figure 3b). 1, 2, 3, 4, 5 [counts the edges diverging from the upper base vertex by pointing at them]; 1, 2, 3, 4, 5 [counts the edges diverging from the lower base vertex]
Yellow	[turns inside the model and counts the edges of a lower base pentagon by pointing at them] 1, 2, 3, 4, 5.
Grey	Look, the base is ten.
Yellow	[counts the middle section] 1, 2, 3, 4... Where did I start? (to Grey) Put your hand here. [continues to count silently] ... [raises arms to the upper base] ten, [lowers arms to the lower base] ten, [makes a circular breaststroke movement with both hands indicating a middle part] ten, ...thirty!

Episode 2 summary

- The focus shifted from a small to a large model
- Large model affords the students:
  - To discern details visually
  - New action (get inside) and new perspective (from the inside)
  - Not all the edges are in the hand reach (from outside)
  - Not all the edges are visible (from the inside)
- Both students perceived useful properties
  - Five edges from each vertex
  - Three edges forming a face
- One of the physical orientations did not allow reasoning based on perceived properties.
- A physical action shifted the focus and structures of attention

Discussion: Shifts in focus and form of students' attention

- By moving in space, changing points of view, and modifying a physical object, the students experienced shifts in focus (small and large model, three distinct parts of the model, vertices, edges, groups of edges) and structures of attention.
- All five theoretical structures of attention and shifts between them (Mason, 2008) were documented in two episodes.
- Shifts in the structures of attention were multimodal: associated with vision, speech, touch, proprioception, and physical actions of students with and through the models.

Discussion: Shifts in focus and form of students' attention

- Tilting the model on its vertex led to shifts in structure of attention and, in turn to an A-Ha moment (c.f. Palatnik & Koichu, 2014, 2015)

- This action helped students answer questions about vertices and edges or explain their solution to their peers in all the cases (more than 20) we possess.

Discussion: The role of physical features of the models in the process of student exploration.

- Models of different scales landed students different affordances for inquiry.
- Each model served as a physical attractor with different affordances for and constraints on the action; accordingly, students reorganized themselves and their exploration in accordance with the affordances and constraints.

Discussion: The role of physical features of the models in the process of student exploration.

- The affordance of a large model to be explored from the inside was realized also in all the cases in our possession.
- Students constructed the models, moved the models, moved between the models and incorporated these actions and features they noticed as a result of actions in their solutions

Discussion

- The case study findings highlight the pedagogical potential of embodied approach in spatial geometry instruction.
- The activity enabled students to ground conceptions of the geometric figure simultaneously as objects in mesospace and mirospace (c.f. Herbst et al., 2017), providing more opportunities for possible shifts in focus and structures of attention and thus learning (Mason, 2008, 2010).
- The fluency with which students moved from one model to another—both physically and inferentially—suggests they noticed invariant scale-free features of a geometric object

Discussion

Students geometric problem solving in the studied activities can be theorized as gradual reconfiguration of their bodily orientation toward the material artifacts, so as to increase their capacity to examine and discuss properties of these artifacts, which they are both generating and transforming

(c.f. Merleau–Ponty, in Dreyfus & Dreyfus, 1999; Hutto, 2019).

Discussion: Processes

- Adjustment (of students' or models) position in space to remove constraints on sensory inspection and manual operation
- Material reconfigurations lead to change in focus and structure of students attention resulting in discovery of new properties and updated foundations for reasoning

Discussion

- The students experienced construction and informal exploration of polyhedron models producing a multitude of perspectives and collaborative insights on their features.
- Their efforts combined collaborative actions, gestures (indexing and iconic), and speech to indicate and highlight models' properties.
- Students' shifts of attention were multimodally grounded in their senses and converged to a gradual disciplinary formalization of the polyhedron's concept

Discussion

An important part of the solution was to communicate a perspective/ a point of view to the peers.

Students actions (e.g. transiting to the model of different size; tilting the model on its vertex) served to improve collective perception by shifting and restructuring peers attention.

Questions

- Is the activity in this case study challenges what we call doing math? Learning math?
- Are the phenomena we have seen in a case study so rare just because “normative” mathematical activities do not allow students to use their senses?
- What methodological tools (multimodal learning analytics) do we need to develop to document and evaluate this type of learning?
- Should we? and how we design embodied activities in a way that creates need for abstraction/ formalization?
- What is the added value of construction of the physical models vs working with a readymade model vs 3D DGE vs XR for learning?

Conclusion

- Constructing and manipulating tangible models creates opportunities for students to harmonize spatial skills and rigorous geometric argumentation as well as bridge iconic and non-iconic visualization.
- The work with 3D models is not straightforward, especially for students who were deprived from multimodal and enactive learning
- School geometry should organize students’ engagement with 3D objects with untampered senses; with gross and fine motor actions; with all the tacit, evolutionarily endowed naturalistic sensibilities for orienting in the environment.
- Once students realize how to work with three-dimensional objects in mathematical activities, they can tap their know-how to build valid arguments grounded in enactive experience.

COMPLEMENTARY SLIDES

The common features

- Spatial geometry tasks;
- Students generate and transform material artifacts
- First construct , then answer questions
- Collaborative problem-solving
- Written instructions with two dimensional diagrams

Still, after over 60 years of empirical research on the use of manipulatives in mathematical classrooms, their cognitive effects and optimal utilization have yet to be established (Bartolini Bussi et al., 2010).

“Movement? There is no need for movement; we are talking about mental growth!” When they think of mental improvement they imagine all are sitting down, moving nothing. But mental development must be connected with movement and is dependent on it. This is the new idea that must enter educational theory and practice…

Watching [the child], one sees that he develops his mind by using his movements. (Montessori, 1949, pp. 203-204)

“One possibility could be to renounce rigorous definitions and undertake to construct a geometry only based on the evidence of empirical space intuition; in this case one should not speak of lines and points, but always only of “stains” and stripes.

The other possibility is to completely leave aside space intuition since it is misleading and to operate only with the abstract relations of pure analysis.

Both possibilities seem to be equally unfruitful: In any case, I myself always advocated the need to maintain a connection between the two directions, once their differences are clear in one’s mind. A wonderful stimulus seems to lay in such a connection. This is why I have always fought in favor of clarifying abstract relations also by reference to empirical models: this is the idea that gave rise to our collection of models in Göttingen.”

(Klein, 2016, p. 221).

Klein, F. (2016). *Elementary mathematics from a higher standpoint: Vol. III. Precision mathematics and approximation mathematics.* Berlin: Springer.

Embodied learning of mathematics

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- Action-Based Embodied Design:
- Using gestures in learning of mathematics
- Developing number sense in young children

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פתרון משימות בגיאומטריה במרחב באמצעות ציור בעט לציור תלת מימדי

- 15תלמידים (ראיון אישי 25; (תלמידים) ראיון קבוצתי 9, קבוצות).
- שרטוט בתלת מימד בהתאם להנחיות של דיאגרמה דו מימדית
- פתרון משימות
- ניתוח תהליך הפתרון

Rosenski, D. & Palatnik, A. (2021, February). *To feel the sketch*. Paper presented at the 9th Jerusalem Conference on Research in Mathematics Education–JCRME9 (in Hebrew).

Rosenski, D. & Palatnik, A. (2022). Secondary students' experience using 3D pen in spatial geometry: affective states while problem solving. In G. Bolondi, F. Ferretti, & C. Spagnolo (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12, February 6 – 10, 2022)*. Bolzano, Italy: ERME.

שרטוט בעט תלת מימדי

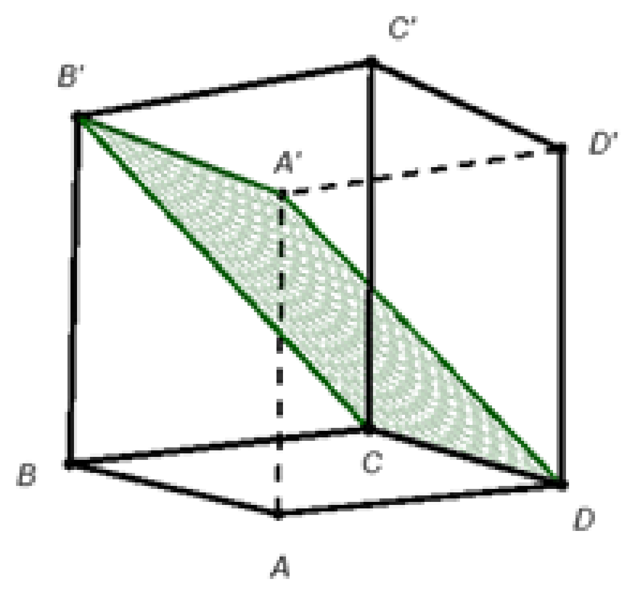
שימוש בעט תלת מימדי

Ng O.L., Sinclair N. (2018) Drawing in Space: Doing Mathematics with 3D Pens. In: Ball L., Drijvers P., Ladel S., Siller HS., Tabach M., Vale C. (eds) *Uses of Technology in Primary and Secondary Mathematics Education*. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-319-76575-4_16

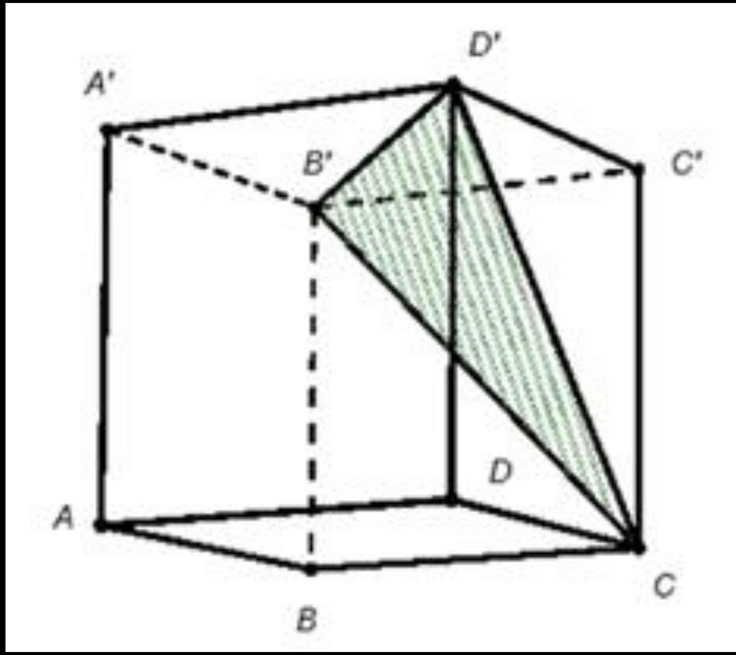
Ng, O.L., Shi, L., Ting, F. (2020). Exploring differences in primary students’ geometry learning outcomes in two technology-enhanced environments: dynamic geometry and 3D printing, *International Journal of STEM Education, 7 (1), 1-13.*

Palatnik, A., (2019).

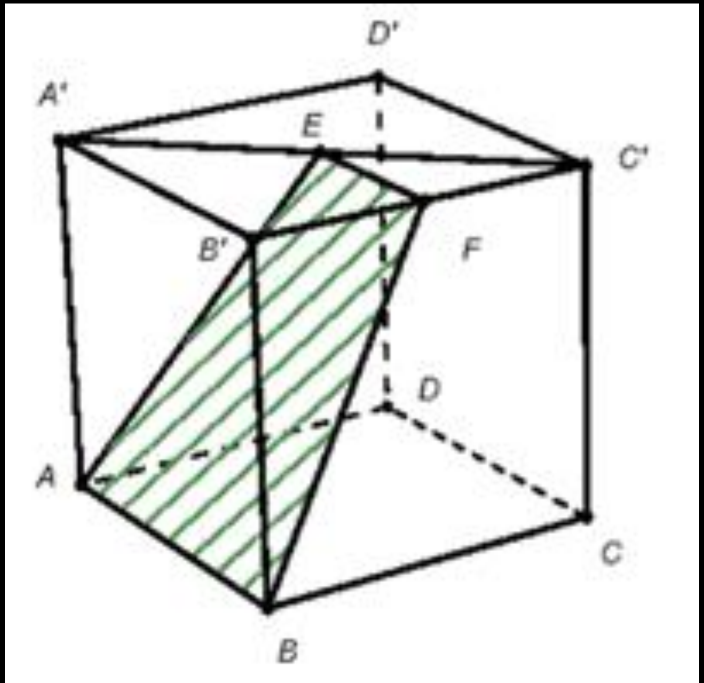
מתודולוגיה –שאלה 1

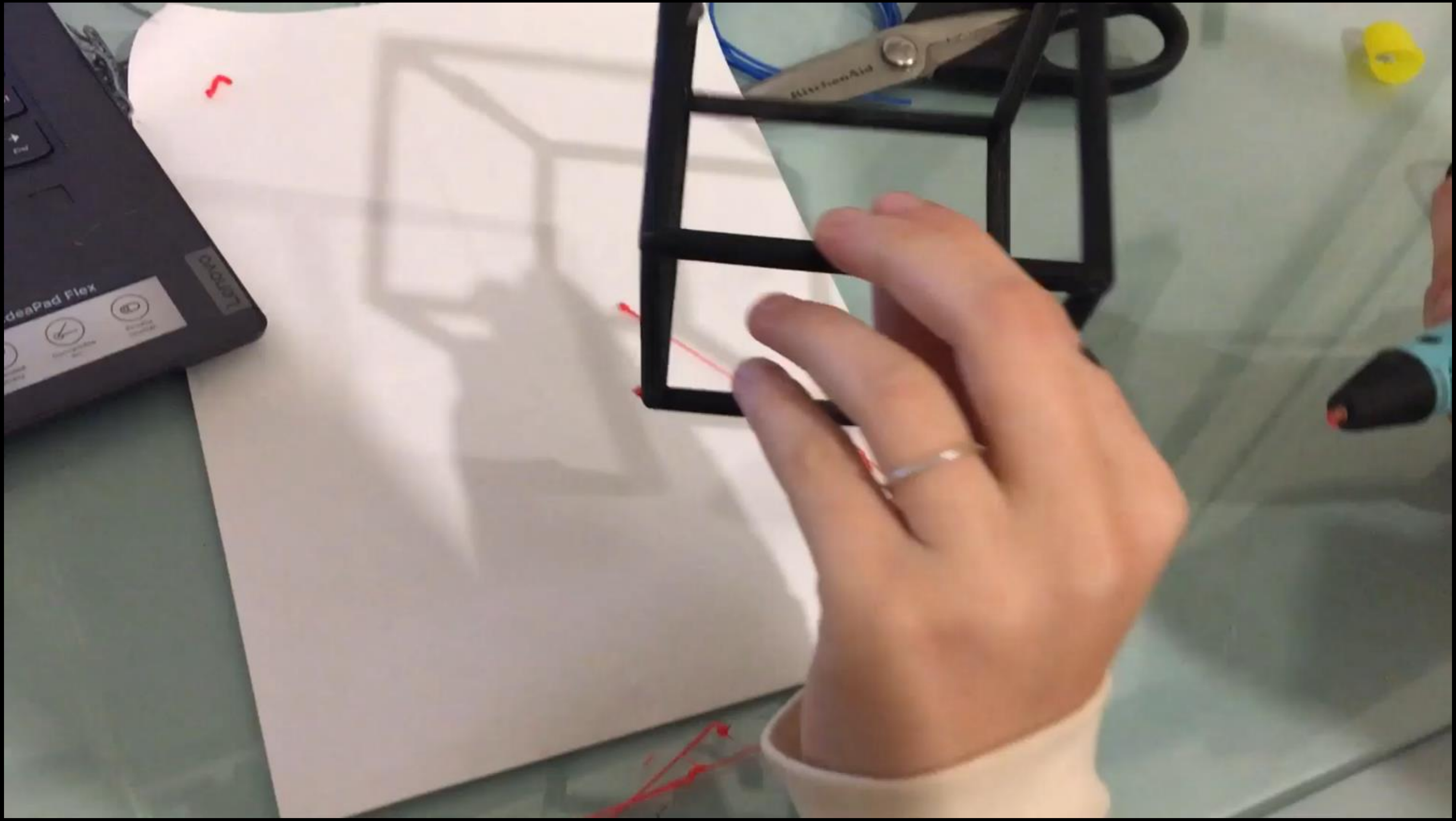


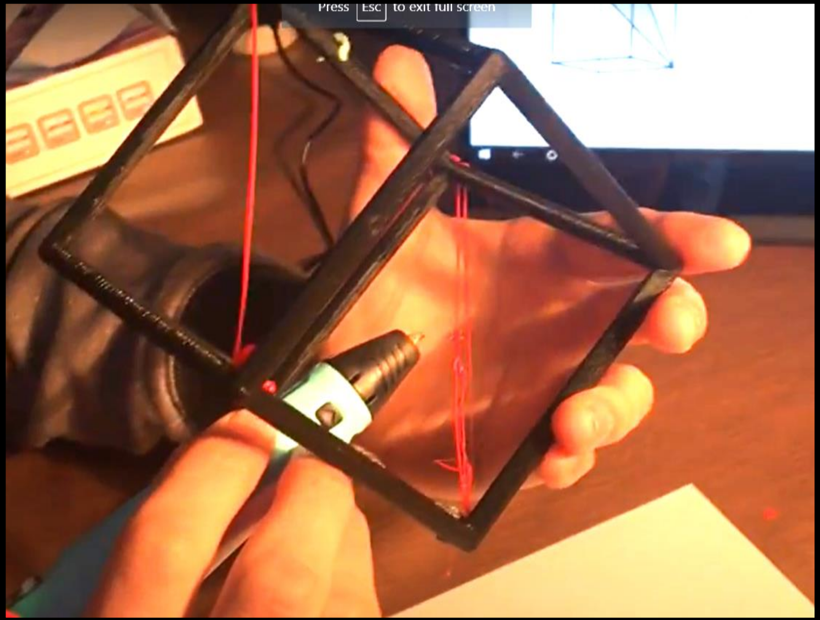
מתודולוגיה –שאלה 2



מתודולוגיה – שאלה 3



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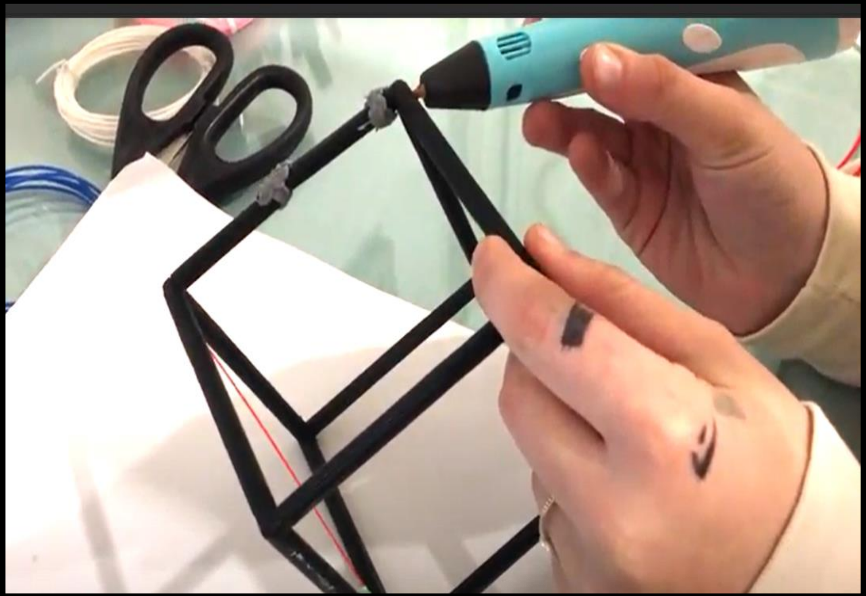
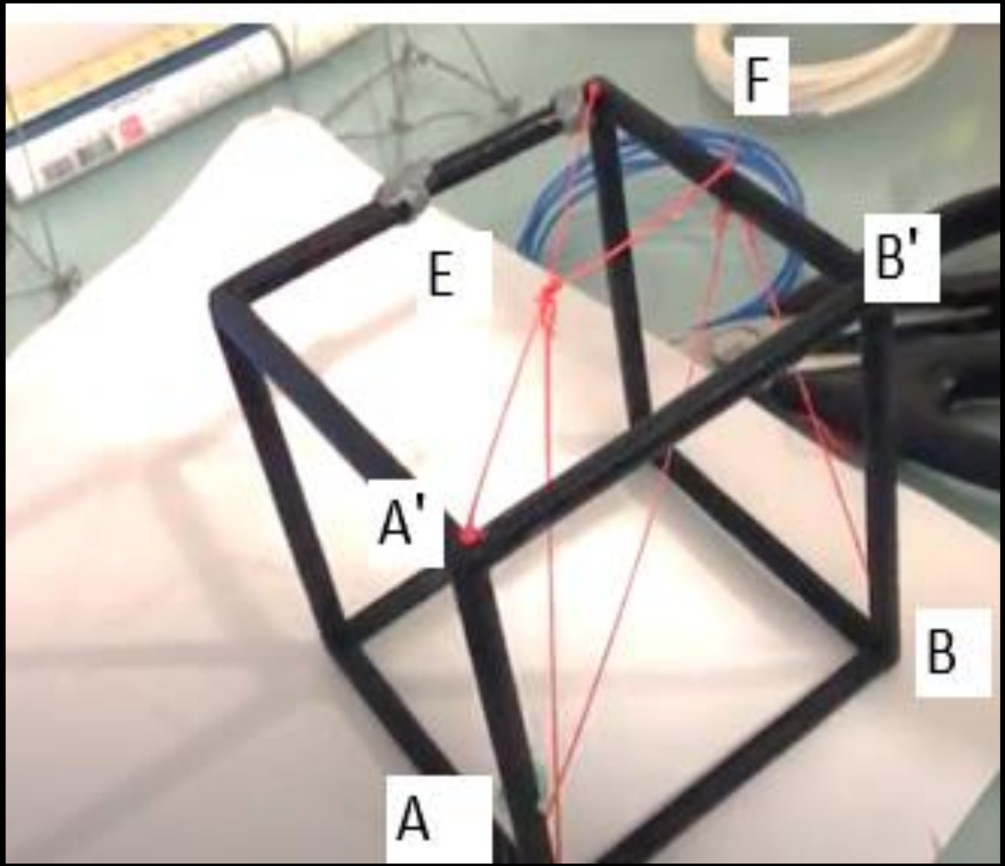
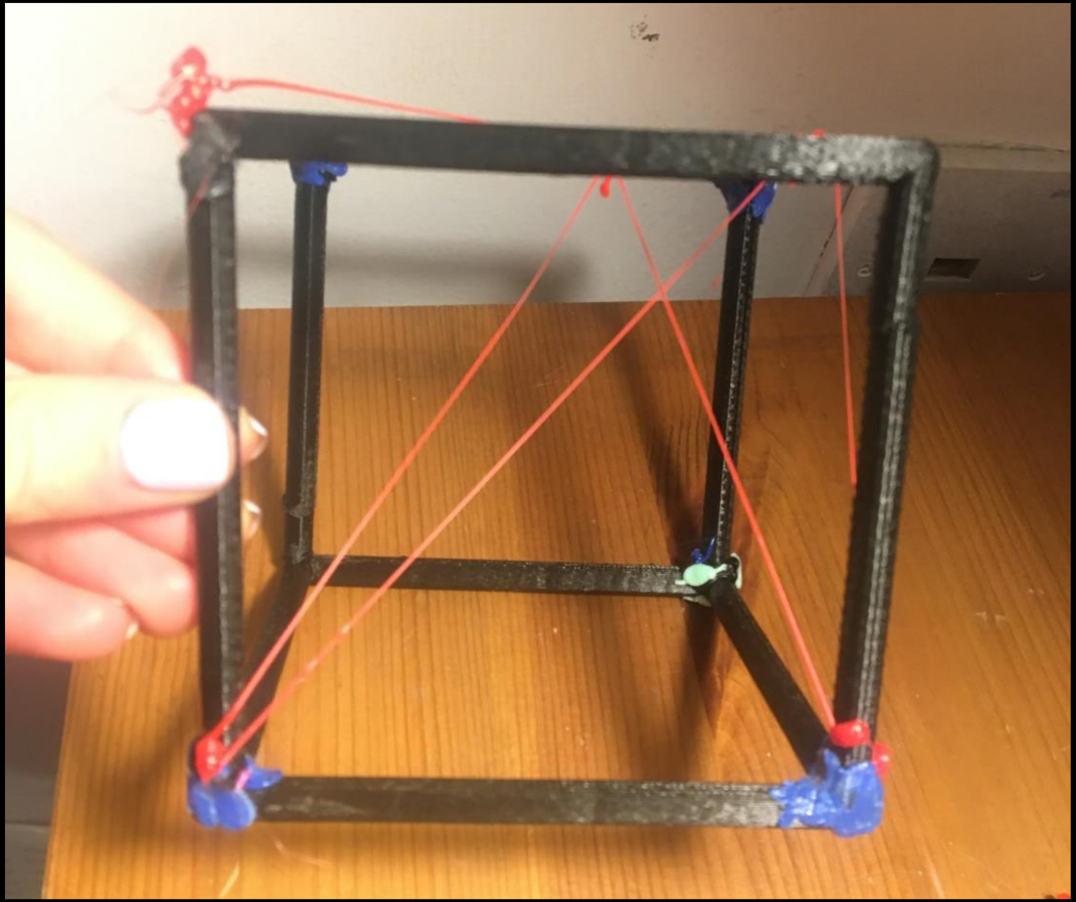


ממצאים -בניית עזר

טרפז -אלכוסניים





- **Is mathematics a rule-like manipulation of abstract symbols that have no direct connections to perception or action?**

Embodied cognition and math

Andres, M., Seron, X., & Olivier, E. (2007). Contribution of hand motor circuits to counting. *Journal of Cognitive Neuroscience, 19*, 563 – 576.

Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition*, *106*(2), 1047-1058.

- There is a connection between hand and number in adults as well. The hand plays a role in mathematical cognition..

- Learning with appropriate gestures is more effective;
- Learning with gestures retantion
  - Mathematical problem solving uses representations based on physical systems of action and perception.It seems reasonable to expect that teaching strategies that take advantage of the body-anchored nature of mathematics will be successful

Molyneux's problem

“Suppose a Man born blind, and now adult, and taught by his touch to distinguish between a Cube, and a Sphere of the same metal, and nighly of the same bigness, so as to tell, when he felt one and t’other, which is the Cube, which the Sphere. Suppose then the Cube and Sphere placed on a Table, and the Blind Man to be made to see. Quære, Whether by his sight, before he touched them, he could now distinguish, and tell, which is the Globe, which the Cube” (Locke, 1694/1975, p.146).

- The answer of Locke is negative: "the blind man, at first sight, would not be able with certainty to say which was the globe, which the cube, whilst he only saw them" (p.146).
- A contemporary neuroscience study (Held, et al., 2011) confirms Locke’s answer.

Thurston, W. P. (1998). How to see 3-manifolds. *Classical and Quantum Gravity*, *15*(9), 2545.

William Paul Thurston

The scale in our imagination can make a big difference in our thinking. An effective strategy is to think about 3-manifolds on the scales we might inhabit: perhaps the size of a house, the size of a stadium, or the size of a town. It is harder to attend as seriously to objects the size you might hold in your hand. It is interesting to sit back and imagine your surroundings—the streets and the land in the neighborhood where you live—and then think of the same degree of imagination about a teacup. At the opposite pole, very huge objects—the size of the universe, or even of the earth—are so far removed from everyday experience that our imagination on those scales tends to be abstract and distant.

Thurston, W. P. (1998). How to see 3-manifolds. *Classical and Quantum Gravity*, *15*(9), 2545.

2. Geometry from the inside

Imagine walking in a barren desert when you see the space in front of you begin to shift. You are startled, and stop. You see a vertical, straight fracture where the left side does not quite match the right: the images overlap ever so slightly. At first you think your vision has gone bad, maybe you have become cross-eyed. However, when you turn your head and move from side to side, the fracture does not turn or move with your head and eyes. When you circle around at a wide distance, you see that the fracture is not fixed on the ground or on the distance scenery, but is localized on a line going straight up into the sky.

Paper models. When you get back to camp, you can make a model to help explain what you have seen. Cut a 350 sector of paper (i.e. a disc with a 10 angle removed) with edges joined to form a blunt cone…

Christian Felix Klein

“A model—whether it be executed and looked at, or only vividly presented—is not a means for this geometry, but the thing itself” (Klein 1893, p. 42).[1]

Halverscheid, S. (2019). Felix Klein’s Mathematical Heritage Seen Through 3D Models. In *The Legacy of Felix Klein* (pp. 131-152). Springer, Cham.

"One possibility could be to renounce rigorous definitions and undertake to construct a geometry only based on the evidence of empirical space intuition; in this case one should not speak of lines and points, but always only of "stains" and stripes.

The other possibility is to completely leave aside space intuition since it is misleading and to operate only with the abstract relations of pure analysis.

Both possibilities seem to be equally unfruitful: In any case, I myself always advocated the need to maintain a connection between the two directions, once their differences are clear in one's mind. A wonderful stimulus seems to lay in such a connection. This is why I have always fought in favour of clarifying abstract relations also by reference to empirical models: this is the idea that gave rise to our collection of models in Göttingen."

(Klein, 2016, p. 221).

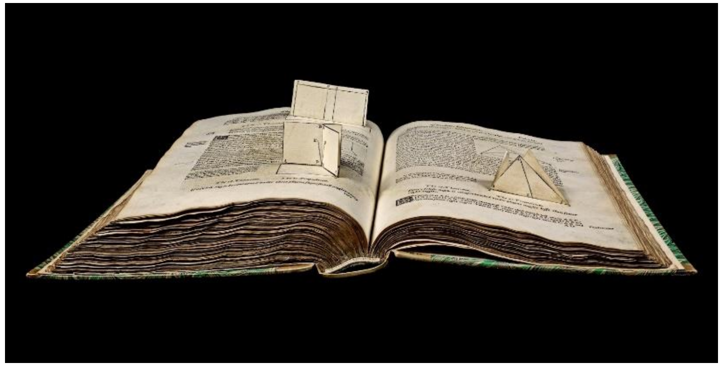
האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

Klein, F. (2016). *Elementary mathematics froma higher standpoint:Vol. III.Precision mathematics and approximation mathematics*. Berlin: Springer.

The collection of models

- http://math.huji.ac.il/~library/models.htm

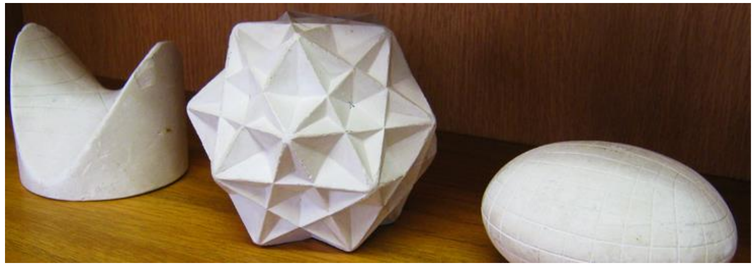
Breaking News!!!
Using manipulatives for learning geometry



John Day’s 1570 pop-up edition of Euclid's Elements

Rousseau, J.-J. (1979). *Emile or on education* (A. Bloom, Trans.). Perseus, Basic Books. (pub. 1755)

Froebel, F. (2005). The education of man (W. N. Hailmann, Trans.). Dover Publications. (pub. 1885).



Felix Klein’s cast models

Montessori, M. (1967). *The absorbent mind*. Holt, Rinehart, and Winston. (pub. 1949).

- Watching a child makes it obvious that the development of his mind comes about through his movements … Mind and movement are parts of the same entity.
- Maria Montessori (1967)

Terence Chi-Shen Tao

Tao recalls the day his aunt found him rolling around her living room floor in Melbourne with his eyes closed.

He was about 23. He was trying to visualize a "mathematical transform". "I was pretending I was the thing being transformed; it did work actually, I got some intuition from doing that."

"Sometimes to understand something you just use whatever tools you have available."

https://www.smh.com.au/lifestyle/terence-tao-the-mozart-of-maths-20150216-13fwcv.html

COMPLEMENTARY SLIDES

The exploration task

A. How many faces does a polyhedron have?
B. How many vertices does the polyhedron you built have? How did you calculate that? Is there another way to know?
C. How many edges does the polyhedron you built have? How did you calculate that? Is there another way to know?
D. Are any parallel edges in a polyhedron? How many?
E. If the icosahedron were half full with water, what would be the shape of the surface of the water when the polyhedron is on a triangular base? What about when the polyhedron is tilted onto a single vertex?
F. What other questions can be invented about this polyhedron?

- Does any model (small or large) have advantages?
- What was easier / harder for you to build?
- What was more fun/sucks to build?
- What was helping/confusing to answer the questions?

“Movement? There is no need for movement; we are talking about mental growth!” When they think of mental improvement they imagine all are sitting down, moving nothing. But mental development must be connected with movement and is dependent on it. This is the new idea that must enter educational theory and practice…

Watching [the child], one sees that he develops his mind by using his movements. (Montessori, 1949, pp. 203-204)

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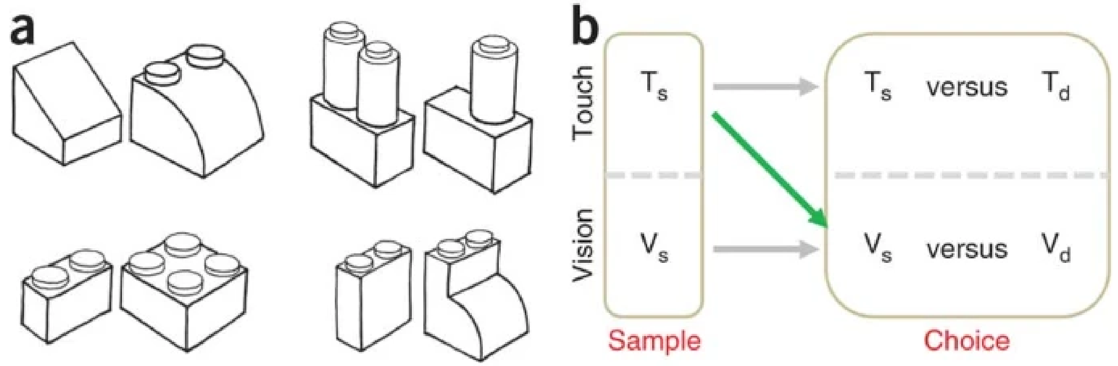
Molyneux's problem

Held, R., Ostrovsky, Y., de Gelder, B., Gandhi, T., Ganesh, S., Mathur, U., & Sinha, P. (2011). The newly sighted fail to match seen with felt. *Nature neuroscience*, *14*(5), 551-553.

Figure 1: Stimuli and testing procedure.



(**a**) Four examples from the set of 20 shape pairs used in our experiments. (**b**) The match-to-sample procedure. The within-modality tactile match to tactile sample task assesses haptic capability and task understanding. The visual match to visual sample task provides a convenient way to assess whether subjects' form vision is sufficient for visually discriminating between test objects. The tactile match to visual sample task represents the critical test of intermodal transfer. T, touch; V, vision; s, sample; d, distractor.